Fano varieties with large pseudoindex

§1. Intro

Setting X : sm. Fano var. of dim $n \geq 4$

$\Updownarrow$ def

X : sm. proj. var. w/ ample $-K_X$

e.g. $\mathbb{P}^n$, Q^n, Rat. Homog., c.i. of low deg.

Fano is one of the outcome of MMP.

$\exists$ Two invariants i_X & ι_X

- $i_X := \max\{m \in \mathbb{Z}_{>0} \mid -K_X = mL \text{ for some } L \in \mathrm{Pic}\, X\}$

  the Fano index of X.

- $\iota_X := \min\{-K_X \cdot C \mid C \subset X : \text{rat. curve}\}$

  the pseudoindex of X

By definition, $\exists$ ample $H \in \mathrm{Pic}\, X$ s.t. $-K_X = i_X H$

$\exists$ rat. curve $C_0 \subset X$ s.t. $\iota_X = -K_X \cdot C_0$

$$\rightsquigarrow \iota_X = -K_X \cdot C_0 = i_X \underset{\substack{\Uparrow \\ \mathbb{Z}_{>0}}}{(H \cdot C_0)} \rightsquigarrow i_X \mid \iota_X$$

Thm ∘ [Kobayashi-Ochiai '73] $i_X \leq n+1$

- $i_X = n+1 \iff X \simeq \mathbb{P}^n$, $i_X = n \iff X \simeq Q^n$
- $i_X = n-1$: del Pezzo var. (Fujita et al. 80's)
- $i_X = n-2$: Mukai var. (Mukai et al. around '90)

Thm · $i_X \le n+1$: Mori theory

- $i_X = n+1 \iff X \simeq \mathbb{P}^n$ (Cho-Miyaoka-Shepherd Barron '02)
- $i_X = n \iff X \simeq Q^n$ (Dedieu-Höring '17)
- $i_X = n-1, n-2$: open. ← WANTED

Mukai Conj. · $\rho_X(i_X-1) \le n$

· "=" $\iff X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$

Bonavero-Casagrande-Debarre-Druel

Generalized Mukai Conjecture · $\rho_X(i_X-1) \le n$

· "=" $\iff X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$

Thm [J. Wiśniewski '90, '91]

(i) $i_X > \frac{n}{2}+1 \Rightarrow \rho_X = 1$

pseudoindex $i_X = \frac{n}{2}+1$ & $\rho_X = 1$ [Occhetta '06]

(ii) $i_X = \frac{n}{2}+1$ & $\rho_X > 1 \Rightarrow X \simeq (\mathbb{P}^{i_X-1})^2$

(iii) $i_X = \frac{n+1}{2}$ & $\rho_X > 1 \Rightarrow X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{i_X}}(2) \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1)^{\oplus i_X-1})$, $\mathbb{P}^{i_X-1} \times Q^{i_X}$ or $\mathbb{P}(T_{\mathbb{P}^{i_X}})$.

↑ $i_X = \frac{n+1}{2}$ & $\rho_X > 1$ WANTED

§2. Main Results

Thm 1

If $i_X = \frac{n+1}{2}$ & $\rho_X > 1$, then X is one of the following:

(i) the bl-up of $\mathbb{P}^n$ along $\mathbb{P}^{i_X-2}$ $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{i_X}}(2) \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1)^{\oplus i_X-1})$

(ii) $\mathbb{P}^{i_X-1} \times Q^{i_X}$

(iii) $\mathbb{P}(T_{\mathbb{P}^{i_X}})$

(iv) $\mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X}$

Thm 2 X : sm. Fano var. w/ $l_X \ge n-2$ & $\rho_X > 1$.
Then X is isomorphic to one of the following:

			n	τ_X	l_X	i_X	ν_X
	(i)	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1))$	4	3	2	1	2
	(ii)	$Bl_{line}(\mathbb{P}^4)$	4	2	2	1	3
	(iii)	a div. on $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(1,1)$	4	2	2	1	3
	(iv)	$\mathbb{P}^2 \times \mathbb{P}^2$	4	$+\infty$	3	3	1
	(v)	$\mathbb{P}^1 \times Q^3$	4	$+\infty$	2	1	1
	(vi)	a Fano 4-fold of $i_X = 2$	4	*	2	2	*
$n \ge 5$	(vii)	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1))$	5	2	2	1	3
	(viii)	$\mathbb{P}^3 \times \mathbb{P}^2$	5	$+\infty$	3	1	1
	(ix)	$Q^3 \times \mathbb{P}^2$	5	$+\infty$	3	3	1
	(x)	$\mathbb{P}(T_{\mathbb{P}^3})$	5	$+\infty$	3	3	1
	(xi)	$\mathbb{P}^3 \times \mathbb{P}^3$	6	$+\infty$	4	4	1

Rem $n-2 \ge \frac{n+1}{2} \iff 2n-4 \ge n+1 \iff n \ge 5$

In Thm 2, the most difficult case is "$n=4$".

Today I focus on Thm 1.

§3. Proof of Thm 1

Setting X: sm. Fano var. of dim $n \geq 4$, $\iota_X = \frac{n+1}{2}$, $\rho_X > 1$.

Key pt.:

Prove that $\exists$ a $\overset{①}{\mathbb{P}^{\frac{n+1}{2}}}$-bdl. str. or $\overset{②}{\mathbb{P}^{\frac{n-1}{2}}}$-bdl str $\pi : X \to W$.

$$\rightsquigarrow \dim W = \frac{n-1}{2} \qquad \rightsquigarrow \dim W = \frac{n+1}{2}$$

$$\xrightarrow[\text{Bonavero-Casagrande-Debarre-Druel}]{} \quad W : \text{Fano w/ } \iota_W \geq \iota_X = \frac{n+1}{2}$$

$$\boxed{\begin{array}{l} \pi : X \to W : \mathbb{P}\text{-bdl} \\ X : \overset{\text{sm.}}{\text{Fano}} \\ \Rightarrow W : \text{sm Fano} \\ \quad \text{w/ } \iota_W \geq \iota_X \end{array}}$$

$\rightsquigarrow$ CMSB, Dedieu-Höring

$$\begin{tikzcd} & \cdot \arrow[dl, "①"'] \arrow[dr, "②"] & \\ W \simeq \mathbb{P}^{\frac{n-1}{2}} & & \mathbb{P}^{\frac{n+1}{2}} \text{ or } Q^{\frac{n+1}{2}} \end{tikzcd}$$

$\mathrm{Br}(W) = 0 \rightsquigarrow \exists \mathcal{E}$: vec. bdl$/W$ s.t. $X = \mathbb{P}_W(\mathcal{E})$.

For $\forall l \subset W = \mathbb{P}^{\frac{n-1}{2}}$: line,

We can prove $\mathcal{E}|_l = \mathcal{O}_{\mathbb{P}^1}^{\oplus \frac{n+3}{2}}$

$\rightsquigarrow \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}}^{\oplus \frac{n+3}{2}}$

$\rightsquigarrow X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{\frac{n-1}{2}}}^{\oplus \frac{n+3}{2}}) \simeq \mathbb{P}^{\frac{n-1}{2}} \times \mathbb{P}^{\frac{n+1}{2}} = \mathbb{P}^{\iota_X - 1} \times \mathbb{P}^{\iota_X}$

Goal Prove that

$\exists$ a $\mathbb{P}^{\frac{n+1}{2}}$-bdl. str. or $\mathbb{P}^{\frac{n-1}{2}}$-bdl str $\pi : X \to W$.

Case 1 $\exists$ bir. elem. contr. $\varphi : X \to Y$

Case 2 $\forall$ elem. contr is of fib. type. ← Today (easy)

Setting X : sm. Fano w/ $i_X = \frac{n+1}{2}$ & $\rho_X > 1$.

Assume $\forall$ elem. contr. of X is of fiber type.

$\exists R \neq R' \subset \overline{NE}(X)$: ext. ray

$$\begin{tikzcd}
 & X \arrow[dl, "\varphi := \mathrm{cont}_R"'] \arrow[dr, "\psi := \mathrm{cont}_{R'}"] & \\
Y & & Z
\end{tikzcd}$$

F : $\forall$ fib of φ

F_{gen} : $\forall$ fib. of φ s.t. $\dim F_{gen} = \dim X - \dim Y$

F' : $\forall$ fib of ψ

F'_{gen} : 〃 ψ s.t. $\dim F'_{gen} = \dim X - \dim Z$

Key Results

X : sm. proj. var.

$\varphi : X \to Y$: contr. of a K_X-neg. ext. ray R, φ : of fiber type.

F: $\forall$ irr. comp. of a non-trivial fiber of φ

$l(R) := \min\{-K_X \cdot C \mid C \subset X : \text{rat. curve} \ \& \ [C] \in R\}$

(i) [Ionescu-Wiśniewski inequality] $\dim F \geq l(R) - 1$

(ii) [Höring-Novelli]

If $\forall$ fiber of φ has dim d & $l(R) = d+1$, then φ is a $\mathbb{P}$-bdl.

$$\left.\begin{array}{l} \dim F \ \geq l(R) - 1 \geqq \frac{n+1}{2} - 1 = \frac{n-1}{2} \\ \dim F' \geq l(R') - 1 \geqq \frac{n-1}{2} \end{array}\right] (*)$$

Since $\dim F + \dim F' - n \leq 0, \quad \text{——} (**)$

$$\dim F, \dim F' \leq \frac{n+1}{2}.$$

$\rightsquigarrow$ $(\dim F_{gen}, \dim \Gamma)$ & $(\dim F'_{gen}, \dim Z)$ are either

$$\left(\frac{n+1}{2}, \frac{n-1}{2}\right) \text{ or } \left(\frac{n-1}{2}, \frac{n+1}{2}\right)$$

Claim φ & ψ are one of the following:

(i) a $\mathbb{P}^{\frac{n+1}{2}}$-bdl;
(ii) a $Q^{\frac{n+1}{2}}$-fib.;
(iii) a $\mathbb{P}^{\frac{n-1}{2}}$-fib.

Pf It is enough to consider the str. of φ.

Assume $\dim F_{gen} = \frac{n+1}{2}$.

By (**),

$$\begin{array}{cccl} \frac{n+1}{2} & & & \\ \shortparallel & & & \\ \dim F_{gen} & & \frac{n-1}{2} & \\ \rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} & \\ \dim F & + & \underline{\dim F'} & - n \leq 0, \quad \text{——} (**) \\ & & \rotatebox{90}{$\geq$} & \\ & & \frac{n+1}{2} & \end{array}$$

$\rightsquigarrow$ $\dim F = \dim F_{gen} = \frac{n+1}{2}$
$\dim F' = \frac{n-1}{2}$.

φ is equidim.

By (*)

$$\frac{n+1}{2} = \dim F \geq l(R) - 1 = \frac{n-1}{2}$$

$$\rightsquigarrow \quad l(R) = \underbrace{\frac{n+3}{2}}_{\text{Höring-Novelli}} \quad \text{or} \quad \underbrace{\frac{n+1}{2}}$$

φ is a $\mathbb{P}^{\frac{n+1}{2}}$ bdl

φ is a $Q^{\frac{n+1}{2}}$-fib.

Assume $\dim F_{gen} = \frac{n-1}{2}$.

By (*) $\quad \frac{n-1}{2} = \dim F_{gen} \geq l(R) - 1 = \frac{n-1}{2}$

$\rightsquigarrow \quad \dim F_{gen} = l(R) - 1 = \frac{n-1}{2}$

$\rightsquigarrow \quad \varphi$ is a $\mathbb{P}^{\frac{n-1}{2}}$-fib. ▨

WLG, we may assume that $\dim F_{gen} \geq \dim F'_{gen}$
(without loss of generality)

Then one of the following holds:

(A) $\varphi : \mathbb{P}^{\frac{n+1}{2}}$-bdl & $\psi : \mathbb{P}^{\frac{n-1}{2}}$-fib.

(B) $\varphi : Q^{\frac{n+1}{2}}$-fib. & $\psi : \mathbb{P}^{\frac{n-1}{2}}$-fib. $\underset{(**)}{\rightsquigarrow}$ $\psi : \mathbb{P}^{\frac{n-1}{2}}$-bdl.

(C) φ & $\psi : \mathbb{P}^{\frac{n-1}{2}}$-fib. $\underset{(**)}{\rightsquigarrow}$ either φ or ψ is a $\mathbb{P}^{\frac{n-1}{2}}$-bdl.